Journal of Algebra **215,** 412–417 (1999)
Article ID jabr.1998.7699, available online at http://www.idealibrary.com on IDEAL®

On a Class of Commutative Power-Associative Nilalgebras*

Ivan Correa and Avelino Suazo†

La Serena, Chile
E-mail: icorrea@elqui.cic.userena.cl, asuazo@elqui.cic.userena.cl

Communicated by E. Kleinfeld

Received November 15, 1997

We prove that commutative power associative nilalgebras of nilindex n and dimension n are nilpotent of index n. We find a necessary and sufficient condition for such an algebra to be a Jordan algebra and give all corresponding isomorphism classes.

1. INTRODUCTION

Gerstenhaber and Myung [1] have proved that commutative power-associative nilalgebras of dimension 4 over a field of characteristic not 2 are nilpotent. Using this result they prove that Suttles's dimension 5 counterexample to Albert's conjecture is the best possible (see Suttles [3]). In particular, Gerstenhaber and Myung prove that a commutative power-associative nilalgebra of nilindex 4 and dimension 4 is nilpotent of index 4.

In this paper we prove that for any n a commutative power-associative nilalgebra of nilindex n and dimension n over a field of characteristic $\neq 2, 3$ is nilpotent of index n.

Gerstenhaber and Myung prove that in the case $n = 4$ there is a $y \in A - A^2$ such that $yA^2 = 0$. We show that the existence of such an element y is precisely the property that characterizes the subclass of Jordan algebras.

* Supported by FONDECYT 1960901/1980548 and DIULS 120-2-27.

† This work forms part of the doctoral research of the second author at Universidad de La Frontera, Temuco, Chile.

0021-8693/99 $30.00

Throughout, A will denote a commutative power-associative nilalgebra of nilindex n and dimension n, $n > 2$ over a field K with $\operatorname{Char} K \neq 2, 3$.

We will denote by $\langle a_1, \ldots, a_j \rangle_K$ the subspace of A generated over K by the elements $a_1, \ldots, a_j \in A$.

Linearizing the identity $x^4 = x^2 x^2$ yields

$$2((xy)x)x + (x^2 y)x + x^3 y = 4(xy)x^2, \tag{1}$$

$$\begin{aligned} &2((yz)x)x + 2((xy)z)x + 2((xy)x)z + 2((xz)y)x + (x^2 y)z \\ &\quad + 2((xz)x)y + (x^2 z)y = 4(yz)x^2 + 8(xy)(xz) \end{aligned} \tag{2}$$

and

$$(xy^2)x + 2((xy)y)x + 2((xy)x)y + (x^2 y)y = 4(xy)^2 + 2x^2 y^2. \tag{3}$$

In what follows we fix an element x in A with $x^{n-1} \neq 0$. We remark that $x^2 \neq 0$, that the powers $x, x^2, \ldots, x^{n-1}$ are linearly independent, and that the subspace $X = \langle x, x^2, \ldots, x^{n-1} \rangle_K$ is an associative subalgebra of A of dimension $n - 1$.

2. NILPOTENCE

Following Gerstenhaber and Myung [1] we obtain the following result:

THEOREM 1. *A is nilpotent of index n.*

Proof. The idea of the proof is to prove that $A^s = X^s$ for any $s > 1$. For $s = 2$ we only need to check that $y^2 \in X^2$ if $y \in A - X$. Let $y \in A - X$ and let $Y = \langle y, y^2, \ldots, y^{n-1} \rangle_K$. We have $n = \dim(X + Y) = n - 1 + \dim Y - \dim X \cap Y$. Hence, since $\dim Y \leq \dim X$, we have

$$\dim X \cap Y = \dim Y - 1 = \dim \langle y^2, \ldots, y^{n-1} \rangle_K < \dim X.$$

Hence, $X \cap Y$ is a proper subalgebra of X and is therefore a subalgebra of X^2. In fact, if we assume the contrary, then there exists an element $a = \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_{n-1} x^{n-1}$ in $X \cap Y$ with $\alpha_1 \neq 0$. Therefore

$$\dim \langle a, a^2, \ldots, a^{n-1} \rangle_K = n - 1 = \dim X$$

which is a contradiction. By the same argument we can prove that $X \cap Y$ is a subalgebra of Y^2. Since $\dim X \cap Y = \dim Y^2$, we get that $Y^2 = X \cap Y$ and therefore $Y^2 \subset X^2$. It follows that $y^2 \in X^2$ and so $A^2 = X^2$.

Since X is an associative subalgebra of A, for all elements $y \in A$ and $k \geq 1$ we have that $((xy)x^k)x = (xy)x^{k+1}, ((xy)x)x^k = (xy)x^{k+1}$, and

$((x^k y)x)x = (x^k y)x^2$, thus if we replace z by x^k in (2) we obtain

$$3x^{k+2}y = 4(xy)x^{k+1} + 2(x^k y)x^2 - 2(x^{k+1}y)x - (x^2 y)x^k. \quad (4)$$

It is now easy to obtain inductively that $x^s y \in X^s$ for all $s \geq 1$ (and therefore X^s is an ideal of A for every $s \geq 1$). We will prove that actually $x^s y \in X^{s+1}$. In fact, since $x^2 y \in X^2$ there are $\alpha_2, \alpha_3, \ldots, \alpha_{n-1} \in K$ such that $x^2 y = \alpha_2 x^2 + \alpha_3 x^3 + \cdots + \alpha_{n-1}x^{n-1}$. If $\alpha_2 \neq 0$ we can assume that $\alpha_2 = 1$. Hence, $x^2(y + \alpha_3 x + \cdots + \alpha_{n-1}x^{n-3}) = x^2$. If we let $z = y + \alpha_3 x + \cdots + \alpha_{n-1}x^{n-3}$ then we have that $x^2 z = x^2$. Then straightforward calculations show that

$$0 = (z + x^2)^n \equiv 2x^2 (\bmod X^4).$$

This is a contradiction. Therefore $\alpha_2 = 0$ and $x^2 y \in X^3$. Using (4) it is easy to obtain inductively that $x^s y \in X^{s+1}$ and $A^s = X^s$ for any $s \geq 2$. In particular $A^n = X^n = 0$, which proves the theorem.

3. JORDAN ALGEBRAS

We remark that we can always find $y \in A - X$ such that $x^2 y = 0$. In fact, if $y' \in A - X$ with $x^2 y' \neq 0$ and if $x^2 y' = \alpha_3 x^3 + \cdots + \alpha_{n-1}x^{n-1}$, then $y = y' - (\alpha_3 x + \cdots + \alpha_{n-1}x^{n-3})$ satisfies $x^2 y = 0$ which proves the claim.

THEOREM 2. *If $n \geq 4$, then A is a Jordan algebra if and only if there is $y \in A - A^2$ such that $yA^2 = 0$.*

Proof. Assume A is a Jordan algebra. Let $y \in A - X$ be such that $x^2 y = 0$. If $xy = \alpha_2 x^2 + \cdots + \alpha_{n-1}x^{n-1}$, we have

$$0 = (x^2 y)x = x^2(yx) = \alpha_2 x^4 + \cdots + \alpha_{n-3}x^{n-1}.$$

It follows that $\alpha_2 = \cdots = \alpha_{n-3} = 0$ and $xy = \alpha_{n-2}x^{n-2} + \alpha_{n-1}x^{n-1}$. Now from (1) we have $x^3 y = 2x^2(xy) = 2\alpha_{n-2}x^n + 2\alpha_{n-1}x^{n+1} = 0$. Replacing z by x^k in (2) we obtain

$$2(x^{k+1}y)x + 3x^{k+2}y = 4(xy)x^{k+1} + 2(x^k y)x^2$$

for any k. Therefore $x^s y = 0$ for all $s \geq 2$, whence $yA^2 = 0$.

We suppose now that there is $y \in A - A^2$ such that $yA^2 = 0$. We will prove that $y \in A - X$. In fact, if $y \in X$ then $y = \alpha_1 x + \cdots + \alpha_{n-1}x^{n-1}$ and $0 = x^2 y = \alpha_1 x^3 + \cdots + \alpha_{n-3}x^{n-1}$. Whence, $\alpha_1 = \cdots = \alpha_{n-3} = 0$ and $y = \alpha_{n-2}x^{n-2} + \alpha_{n-1}x^{n-1} \in A^2$, which is a contradiction. Therefore $y \in A - X$.

Now from (1) we obtain $2(xy)x^2 = 0$ and therefore $xy \in X^{n-2}$. From (3) we obtain $x^2y^2 = 0$ and therefore $y^2 \in X^{n-2}$.

If $x_0 = \alpha_0 y + p$ and $y_0 = \beta_0 y + q$ with $p, q \in X$ arbitrary elements in A, we have $(x_0^2 y_0)x_0 - x_0^2(y_0 x_0) = (p^2q)p - p^2(qp)$. Since X is an associative subalgebra of A the result follows. This proves the theorem.

COROLLARY. *If n is 3 or 4 then A is a Jordan algebra if and only if it is a power-associative algebra.*

Proof. It is known that a commutative nilalgebra of nilindex 3 is a Jordan algebra and that any Jordan algebra is power-associative (see [4, 2]). Let $n = 4$. If A is a power-associative algebra and $y \in A - X$ is such that $x^2y = 0$, then, since $x^3y \in X^4 = 0$ we obtain that A is a Jordan algebra from Theorem 2. This proves the corollary.

Dimension 4 is the highest dimension for which the Corollary is valid, as the following example shows:

EXAMPLE. Let B be a commutative algebra with basis y, x, x^2, x^3, x^4 and nonzero multiplication given by $xy = x^2$, $x^3y = 2x^4$, and $x^ix^j = x^{i+j}$ for $2 \le i + j \le 4$. B is a power-associative nilalgebra of nilindex 5 which is not Jordan since $0 = (x^2y)x \neq x^2(xy) = x^4$.

4. CLASSIFICATION

Throughout this section the dimension will be always assumed to be ≥ 4. If $y \in A - A^2$ is such that $yA^2 = 0$, then from the proof of the Theorem 2 we conclude that $xy = \alpha x^{n-2} + \delta x^{n-1}$ and $y^2 = \beta x^{n-2} + \gamma x^{n-1}$. If we let $y' = y - \delta x^{n-2}$, then we have $xy' = xy - \delta x^{n-1} = \alpha x^{n-2}$ and $y'^2 = y^2 = \beta x^{n-2} + \gamma x^{n-1}$. Hence we can assume that $\delta = 0$. In order to obtain a classification theorem for these algebras it is only necessary to know the products xy and y^2.

For Theorem 3 we will need additionally that Char K not be divisor of $n - 2$.

THEOREM 3. *If A is a Jordan algebra, then there is $y \in A - A^2$ such that $yA^2 = 0$, and the products xy and y^2 are given by one of the following list*:

(I) $xy = 0$, $y^2 = 0$.

(II) $xy = 0$ *and* $y^2 = \gamma x^{n-1}$ *with* $\gamma \neq 0$.

(III) $xy = 0$ *and* $y^2 = \beta x^{n-2}$ *with* $\beta \neq 0$.

(IV) $xy = x^{n-2}$ *and* $y^2 = 0$.

(V) $xy = x^{n-2}$ *and* $y^2 = \gamma x^{n-1}$ *with* $\gamma \neq 0$.

(VI) *If* $n = 4$, $xy = x^2 = y^2$.

Proof. Let $xy = \alpha x^{n-2}$ and $y^2 = \beta x^{n-2} + \gamma x^{n-1}$.

Case $\alpha = 0$. If $\beta = \gamma = 0$ we obtain the algebra (I). If $\beta = 0$ and $\gamma \neq 0$ we obtain (II). If $\beta \neq 0$, we have $y^2 = \beta(x + (\gamma/\beta(n-2))x^2)^{n-2}$. Therefore, letting $x' = x + (\gamma/\beta(n-2))x^2$ we obtain $x'y = 0$ and $y^2 = \beta x'^{n-2}$. Hence we may assume $\gamma = 0$, which gives (III).

Case $\alpha \neq 0$. If $y' = (1/\alpha)y$ we have that $xy' = x^{n-2}$ and $y'^2 = \beta' x^{n-2} + \gamma' x^{n-1}$. We may assume that $\alpha = 1$. Therefore $xy = x^{n-2}$ and $y^2 = \beta x^{n-2} + \gamma x^{n-1}$. If $y^2 = 0$ we obtain (IV). If $\beta = 0$ and $\gamma \neq 0$ we obtain (V). If $\beta \neq 0$, let $x' = x + (\gamma/\beta(n-2))x^2$ and $y' = y + (\gamma/\beta)x^{n-2}$. Then $y'^2 = \beta x'^{n-2}$ and $x'y' = x'^{n-2}$. Consequently, we may assume that $\gamma = 0$. Thus $xy = x^{n-2}$ and $y^2 = \beta x^{n-2}$ with $\beta \neq 0$. We note that $y(y - \beta x) = 0, (y - \beta x)^3 = -\beta^3 x^3 + \beta^2 x^{n-1}$, and for all $k \geq 4$, $(y - \beta x)^k = (-1)^k(\beta x)^k$. Hence for $n \geq 5, (y - \beta x)^{n-1} = (-1)^{n-1}\beta^{n-1}x^{n-1} \neq 0$. In such case, if $x' = y - \beta x$, we have $x'y = 0$ and the algebra is one of the above.

Now, if $n = 4$, we have $(y - \beta x)^3 = \beta^2(1 - \beta)x^3$. If $\beta^2(1 - \beta) \neq 0$, the algebra is one of the above. If $\beta^2(1 - \beta) = 0$, then $\beta = 1$ and $y^2 = xy = x^2$, and we obtain (VI).

If $C = \{a \in A\colon aA^2 = 0\} = Ky + A^{n-2}$, then $C^2 = 0$ in cases (I) and (IV), $0 \neq C^2 \neq A^{n-1}$ in (III) and (VI), and $C^2 = A^{n-1}$ in (II) and (V), and also $CA = A^{n-1}$ in (I) and (II) and $CA = A^{n-2}$ otherwise. Hence only the cases (III) and (VI) have to be separated in dimension 4. But in (III), $\{a \in A\colon aC \subset A^{n-1}\} = X$, where there are elements of order n, and in (VI) $\{a \in A\colon aC \subset A^{n-1}\} = K(x - y) + A^2$, where any element has order ≤ 3. This shows that algebras from different classes (I), (II), ..., (VI) are not isomorphic.

We observe that if y and y' are elements in $A - A^2$ with $yA^2 = 0$, $y'A^2 = 0$, then $C = Ky + A^{n-2} = Ky' + A^{n-2}$. Thus $y' = \delta y + \varepsilon x^{n-2} + \lambda x^{n-1}$, where $\delta \neq 0$. Therefore $y'^2 = \delta^2 y^2$.

Now, we will obtain the isomorphism classes in cases (II), (III), and (V). In the following we will assume that $n \geq 5$. The case $n = 4$ was covered by Gerstenhaber and Myung in [1].

Let B be another commutative Jordan nilalgebra of nilindex n and dimension n over K, and $y_0, x_0, x_0^2, \ldots, x_0^{n-1}$ a basis of B such that $y_0 \in B - B^2$, $y_0 B^2 = 0$, $x_0 y_0 = \alpha_0 x_0^{n-2}$, and $y_0^2 = \beta_0 x_0^{n-2} + \gamma_0 x_0^{n-1}$.

We observe that if $\varphi\colon B \to A$ is an isomorphism, then $\varphi(y_0) \in A - A^2$ and $\varphi(y_0)A^2 = 0$. Hence $\varphi(y_0) = \delta y + \varepsilon x^{n-2} + \lambda x^{n-1}$ where $\delta, \varepsilon, \lambda \in K$ and $\delta \neq 0$. Hence, $\varphi(y_0^2) = \varphi(y_0)^2 = \delta^2 y^2$. Now, if $\varphi(x_0) = \alpha_0 y + \alpha_1 x + \cdots + \alpha_{n-1}x^{n-1}$ then we have that $\varphi(x_0^{n-2}) = \alpha_1 x^{n-2} + F(\alpha_0, \alpha_1, \alpha_2)x^{n-1}$ and $\varphi(x_0^{n-1}) = \alpha_1^{n-1}x^{n-1}$, where $F(\alpha_0, \alpha_1, \alpha_2)$ is a certain polynomial expression in $\alpha_0, \alpha_1, \alpha_2$.

Case (II). If n is an even number, letting $x' = \gamma x$ and $y' = \gamma^{(n-2)/2} y$, we obtain $x'y' = 0$ and $y'^2 = x'^{n-1}$. So in this case there is a unique algebra. If n is an odd number and $\gamma_0 \neq 0 \in K$, $x_0 y_0 = 0$, and $y_0^2 = \gamma_0 x_0^{n-1}$ we have that if $\varphi\colon B \to A$ is an isomorphism, then $\varphi(y_0^2) = \delta^2 y^2$. Hence we obtain $\delta^2\gamma = \gamma_0 \alpha_1^{n-1}$, that is, $\gamma_0/\gamma \in (k^*)^2$. Conversely, if $\gamma_0/\gamma = k^2 \in (k^*)^2$ the linear function $\varphi\colon B \to A$ such that $\varphi(y_0) = ky$ and $\varphi(x_0^i) = x^i$ for $i = 1, 2, \dots, n$ is an isomorphism of algebras. Thus in this case we have found one family of algebras parametrized by $k^*/(k^*)^2$.

Case (III). If n is an odd number, letting $x' = \beta x$ and $y' = \beta^{(n-3)/2} y$, we obtain $x'y' = 0$ and $y'^2 = x'^{n-2}$. So in this case we have again a unique algebra. Now, if n is an even number, $\beta_0 \neq 0$, $x_0 y_0 = 0$, $y_0^2 = \beta_0 x_0^{n-2}$, and $\varphi\colon B \to A$ is an isomorphism, then we have $\delta^2\beta = \beta_0 \alpha_1^{n-2}$. Hence $\beta_0/\beta \in (k^*)^2$. Conversely, if $\beta_0/\beta = k^2 \in (k^*)^2$, the linear function $\varphi\colon B \to A$ such that $\varphi(y_0) = ky$, $\varphi(x_0^i) = x^i$, $i = 1, 2, \dots, n$, is an isomorphism. Thus in this case we have found another family of algebras parametrized by $k^*/(k^*)^2$.

Case (V). Let $\gamma_0 \neq 0$, $x_0 y_0 = x_0^{n-2}$, and $y_0^2 = \gamma_0 x_0^{n-1}$. If $\varphi\colon B \to A$ is an isomorphism we obtain $\delta^2\gamma = \gamma_0 \alpha_1^{n-1}$. Since $\varphi(x_0 y_0) = \varphi(x_0)\varphi(y_0)$, we have $\alpha_1 \delta = \alpha_1^{n-2}$. Thus $\delta = \alpha_1^{n-3}$. Therefore $\alpha_1^{n-5}\gamma = \gamma_0$. Now, if $n = 5$, A and B are isomorphic if and only if $\gamma = \gamma_0$. If $n > 5$, and there is a $k \in K$ such that $k^{n-5}\gamma = \gamma_0$, then the linear function $\varphi\colon B \to A$ such that $\varphi(y_0) = k^{n-3} y$, $\varphi(x_0^i) = (kx)^i$, $i = 1, 2, \dots, n$ is an isomorphism of algebras. This completes our classification.

ACKNOWLEDGMENTS

The authors thank the referee for many useful remarks.

REFERENCES

1. M. Gerstenhaber and H. C. Myung, On commutative power-associative nilalgebras of low dimension, *Proc. Amer. Math. Soc.* **48** (1975), 29–32.
2. R. D. Schafer, "An Introduction to Nonassociative Algebras," Academic Press, New York/London, 1966.
3. D. Suttles, A counterexample to a conjecture of Albert, *Notices Amer. Math. Soc.* **19** (1972), A-566.
4. K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov, and A. I. Shirshov, "Rings That Are Nearly Associative," Academic Press, New York/London, 1992.